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QUANTUM CALCULUS AND ITS APPLICATION TO FRACTIONAL DIFFERENTIAL EQUATIONS

ABSTRACT
of thesis for the degree of Doctor of Philosophy (PhD) the specialty 6D060100 –«Mathematics»

Relevance of the topic. The dissertation work is devoted to quantum calculus and its applications in differential equations with fractional derivatives.

In mathematics, quantum calculus, sometimes called unrestricted calculus, is equivalent to traditional infinitesimal calculus without the concept of limits. It defines q-calculus and h-calculus. The development of these two branches originates from the studies of P. Cheng and V. Kas at the beginning of the 20th century.

At the beginning of the 18th century, L. Euler proposed the most common language of quantum calculus - q-calculus. In 1748, he considered the infinite product of the form $(q;q)_\infty^{-1} = \prod_{k=0}^{\infty} \frac{1}{1-q^{k+1}}, |q|<1$ as a generating function for $p(n)$. In addition, he discovered the first two q-exponential functions; this in turn was the premise of the q-binomial theorem. A hundred years later, this research process was continued by E. Hein.

In 1987, A. Lupas first began to use the q-calculus in the field of approximation theory. He was the first to introduce q-Bernstein polynomials, and these studies were widely developed. Important information about the research results is contained in the book. We also especially note the work of T. Ernst. This source mentions many applications of the q-calculus in oscillation theory, interpolation theory, quantum groups, quantum algebras, hyper geometric series, complex analysis, and elementary particle physics.

There is considerable interest in these topics today, and the q-calculus has served as a bridge between mathematics and physics for the past two decades. q-calculus has many applications in various areas of mathematics such as dynamical systems, number theory, combinatory, special functions, fractals, and is widely used in scientific problems in some applied areas such as computer science, quantum mechanics, and other quantum physics. Most of the additional information can be found in the work of J. Gasper and M. Rahman, which contains simple proofs of many results (for example, Clausen's q-formula, q-orthogonal polynomials, q-analogues of various multiplication formulas, etc.) and important applications in other areas (for example, modern algebra, real and complex analysis, number theory, etc.).

In the past three decades, fractional differential equations have attracted a lot of attention and are widely used in phenomena related to physics, chemistry, biology, signal and image processing, as well as food additives, weather and economics, etc. include social fields. Thus, ordinary differential equations and differential equations with fractional derivatives have received significant development, a large number of articles and several books have been published on this topic in various fields, for example, T. Sandev and Z. Tomovsky, A.A. Kilbas et al., R. Hilfer, monographs by K.S. Miller and B. Rossa and references therein.

Fractional calculus is one of the branches of mathematics that studies integration and differentiation of a real or complex order. Fractional-order equations based on the Riemann-Liouville and Caputo fractional derivative require initial conditions for integer-order equations. Accordingly, equations with fractional derivatives have attracted the interest of researchers in various fields.

The origin of q-difference calculus can be found in the works of F. Jackson and R.D. Carmichael, this is the beginning of the twentieth century. Recently U. Alsalam and R.P. Agarwal proposed a q-difference fractional calculus. Today, perhaps due to the rapid growth of research in the field of fractional q-differential calculus, new developments in the theory of fractional q-differential calculus have been reviewed in detail by several researchers. For example, some researchers have obtained q-analogues of the properties of integral and differential fractional operators.

It should be noted that so far much attention has been paid to q-differential equations. Several papers have been published on the existence, singularity, or multiplicity of solutions of non-linear q-differential fractional equations using some well-known fixed point theorems.

However, the theory of q-differential equations with constant and variable coefficients is still in its infancy, and many aspects of this theory still require research. As is known, the theory of the Cauchy problem for linear, homogeneous and inhomogeneous differential equations, based on the main fractional Caputo derivative, is still under development.

Therefore, the use of quantum calculus, including q-calculus, in finding solutions to differential equations with fractional derivatives is relevant.

The purpose of the study. Application of 1-calculus in quantum calculus to equations with fractional derivatives, finding their solutions and proving their existence and uniqueness.

Research objectives: To achieve the main goal, it is necessary to solve the following problems:

- to prove the equivalence of a nonlinear problem of Cauchy type with a fractional Riemann-Liouville q-derivative and a q-integral Volterra equation. Based on this theorem, to obtain the existence and uniqueness of a unique solution of the Cauchy-type problem in space;
- to obtain the existence and uniqueness of a solution to a Cauchy-type problem of a linear Cauchy-type problem with a fractional Caputo q-derivative ${}^{c}D_{q,0+}^{\alpha} f$ of order $\alpha > 0$;

- to obtain a q-analogue of the q-fractional Hilfer derivative. Proof of the equivalence of the q-integral Volterra equation of a nonlinear Cauchy-type problem with a q-fractional Hilfer derivative. Based on this theorem, obtaining the existence and uniqueness of a unique solution of a Cauchy-type problem in space $L^1_{\alpha,\beta,q}[a,b]$;
- To obtain exact solutions of a new modification of the Schrödinger equation obtained using the q-Bessel operator. Proof of the existence and uniqueness of this solution in a space of Sobolev type $W_q^2\left(R_q^+\right)$.

The objects of research. q-differential equations with fractional derivatives. Cauchy-type linear problem with q-fractional Riemann-Liouville derivative. Nonlinear Cauchy type problem with q-fractional Riemann-Liouville derivative. Cauchy-type linear problem with q-fractional Caputo derivative. The Schrödinger equation given by the q Bessel operator. Nonlinear fractional q-differential equations with q-fractional derivative of Hilfer type.

Research methods. The dissertation uses the method of successive approximations to construct a solution to a q-differential equation with a fractional derivative in quantum calculus and determine its uniqueness. By applying the q-Bessel Fourier transform to the Cauchy-type problem, methods of transition to ordinary differential equations were used.

Scientific novelty. Solutions of fractional differential equations in quantum calculus using Riemann-Liouville, Caputo and Hilfer fractional derivatives.

Results presented to the defense.

- An equivalence theorem for a nonlinear Cauchy-type problem with a q-fractional Riemann-Liouville derivative and a q-integral Volterra equation is proved, on the basis of which the existence and uniqueness of a unique solution of the Cauchy-type problem in the space $L^1_{\alpha,q}[a,b]$ are proved;

- The existence and uniqueness of a solution to a certain linear problem of Cauchy type with a q-fractional Riemann-Liouville derivative is proved;

- The existence and uniqueness of a solution to a linear problem of Cauchy type with a q-fractional Caputo derivative ${}^cD^{\alpha}_{q,0+}f$ of order $\alpha > 0$ is proved;

- Exact solutions of a new modification of the Schrödinger equation are obtained using the Bessel q-operator. The existence and uniqueness of this solution in a space of Sobolev type $W_q^2\left(R_q^+\right)$ is proved;

- A q-analogue of the fractional Hilfer derivative is obtained.

- An equivalence theorem for the q-integral Volterra equation of a nonlinear Cauchy-type problem with a q-fractional Hilfer derivative is proved. On the basis of this theorem, the existence and uniqueness of a unique solution of the Cauchy-type problem in the space 3 is obtained.

The theoretical and practical value of the results. This research is largely fundamental and will make a great contribution to the development of quantum calculus in equations with fractional derivatives.

Personal contribution of the applicant The research work presented in the dissertation was carried out with the direct participation of the author. A non-linear problem of the Cauchy type of the q -fractional Riemann-Liouville derivative and its equivalence to the Volterra integral equation are proved, theorems on the existence and uniqueness of the solution are proved by the method of successive approximations. A new q -fractional derivative analogous to the Hilfer fractional q -derivative is obtained, its equivalence to the Volterra integral equation is proved, existence and uniqueness theorems are proved by the method of successive approximations. The results obtained were published in the form of scientific articles and scientific theses.

Approbation of the dissertation results. The main results of the work presented at the:

- XV International scientific conference of students and young scientists «ǴYLYM JÁNE BILIM – 2020» (Nur-Sultan**,** 2020);
- Traditional international April mathematical conference in honor of the Day of Science Workers of the Republic of Kazakhstan, dedicated to the 1150th anniversary of Abu Nasyr al-Farabi and the 75th anniversary of the Institute of Mathematics and Mathematical Modeling (Almaty, 2020);
- XVI International scientific conference of students and young scientists «ǴYLYM JÁNE BILIM – 2021» (Nur-Sultan**,** 2021);
- 1 International scientific and practical conference «IMPORTANCE OF SOFT SKILLS FOR LIFE AND SCIENTIFIC SUCCESS» (Ukraine, 2022).

Individual results of the dissertation:

- performed at the scientific seminar "Functional analysis and its application" (supervisors of the seminar were academicians of NAS RK M.Otelbaev and R. Oinarov, professors E.D. Nursultanov, K.N. Ospanov);
- discussed on the Scientific seminars at Luleå Technical University and at the University of Tromsø – The Arctic University of Norway, Faculty of Mathematics, led by Professor L.E. Persson;
- presented and discussed at the scientific seminar "Weighted inequalities and their applications" (supervisors of the seminar were academician of NAS RK R. Oinarov, associate professors A.M. Temirkhanova, A.M.Abylayeva, associate professor M. Alday).

Publications. 8 works on the topic of the dissertation, including 3 articles in scientific publications included in the list recommended by the Control Committee for Education and Science of the MES RK, 1 article indexed in the database Scopus, Web of Science (Web of Science, Impact factor – 1.25, 2020, Q2) 4 publications in the materials of international scientific conferences, including 1 publication in the materials of foreign international conferences.

The structure and scope of the dissertation. The dissertation consists of an introduction, three chapters, a conclusion and a list of references. The total volume of the dissertation is 76 pages.

The first chapter contains all the formulas, definitions and lemmas needed to prove the theorems of the second and third chapters.

In the second chapter, a nonlinear problem of the Cauchy type for the Riemann-Liouville equation with a q-fractional derivative is considered. An equivalence theorem is proved, the existence and uniqueness of a solution to a Cauchy-type problem in a given space is proved. A q-analogue of the Hilfer derivative operator is also defined. Equivalence theorems are proved for the q-fractional problem of Cauchy type and the q-integral Volterra equation.

In the third chapter, exact and numerical solutions of linear-fractional q-differential equations and Cauchy-type problems related to the Riemann-Liouville fractional q-derivative in q-calculus were considered. In addition, exact solutions of linear-fractional q-differential equations with a q-fractional Caputo derivative of order $\alpha > 0$ are created. In addition, exact solutions of a new modification of the Schrödinger equation related to the Bessel q-operator are obtained. An existence theorem for this solution in a Sobolev-type space in q-calculus is proved.

In the conclusion, the main conclusions are formulated and the scope of their application is described. The dissertation ends with a list of used literature.